Lecture 25 04/25/2018

Radiation (Cont'd)

Now, we consider the $\ell=1$ term in the expansion for the vector potential.

For $r>r'$, we have:

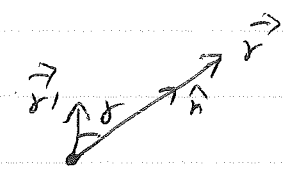
$$\vec{A}(\vec{x}) = \mu_0 ik\, h_1^{(1)}(kr) \int \vec{J}(\vec{x}')\, j_1(kr') \sum_{m=-1}^{+1} Y_{1m}(\theta,\phi)\, Y_{1m}^*(\theta',\phi')$$

In the long wavelength approximation, wher $kr' \ll 1$, we have:

$$j_1(kr') \simeq \frac{kr'}{3}$$

Also, note that:

$$\sum_{m=-1}^{+1} Y_{1m}(\theta,\phi)\, Y_{1m}^*(\theta',\phi') = \frac{3}{4\pi} P_1(\cos\gamma)$$



Thus:

$$\vec{A}(\vec{x}) \simeq \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right) \int \vec{J}(\vec{x}')\,(\hat{n}\cdot\vec{x}')\, d^3x'$$

We can write the integral as:

$$n_j \hat{e}_i \int J_i(\vec{x}')\, x'_j\, d^3x' = n_j \hat{e}_i \left\{ \frac{1}{2}\int \left[x'_j J_i(\vec{x}') - x'_i J_j(\vec{x}')\right] d^3x' + \frac{1}{2}\int \left[x'_j J_i(\vec{x}') + x'_i J_j(\vec{x}')\right] d^3x' \right\}$$

The first integral is:

$$n_j \hat{e}_i \propto \frac{1}{2} \varepsilon_{ijk} \int [\vec{x}' \times \vec{J}_{(\vec{x}')}]_k \, d^3x' = \vec{m} \times \hat{n}$$

Where $\vec{m}$ is the magnetic dipole moment of the source (the physical magnetic dipole moment is $Re(\vec{m}e^{-i\omega t})$). The magnetic dipole contribution to the vector potential is:

$$\vec{A}_m(\vec{x}) \simeq \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \left(\frac{1}{r} - ik\right) \vec{m} \times \hat{n} \underset{\substack{kr \gg 1 \\ \text{limit}}}{\simeq} ik \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} (\hat{n} \times \vec{m})$$

Then (in the far-field region $kr \gg 1$):

$$\vec{B}_m(\vec{x}) = \vec{\nabla} \times \vec{A}_m(\vec{x}) \simeq \frac{k^2 \mu_0}{4\pi} \frac{e^{ikr}}{r} (\hat{n} \times \vec{m}) \times \hat{n}$$

$$\vec{E}_m(\vec{x}) = \frac{-1}{\mu_0 \varepsilon_0 i\omega} \vec{\nabla} \times \vec{B}_m(\vec{x}) \simeq \frac{\mu_0 c k^2}{4\pi} \frac{e^{ikr}}{r} ((\hat{n} \times \vec{m}) \times \hat{n}) \times \hat{n} \underset{\hat{n}\cdot\hat{n} = 1 \text{ used}}{=} \frac{-\mu_0 c k^2}{4\pi} \frac{e^{ikr}}{r} (\hat{n} \times \vec{m})$$

This results in:

$$\frac{dP_m}{d\Omega} = \frac{1}{2\mu_0} Re \left[\vec{E}_m(\vec{x}) \times \vec{B}_m^*(\vec{x})\right] \cdot \hat{n} \, r^2 \underset{(\hat{n} \times \vec{m})\cdot\hat{n} = 0 \text{ used}}{=} \frac{\mu_0 c k^4}{32\pi^2} |\hat{n} \times \vec{m}|^2$$

We see that all considerations of electric dipole radiation apply here upon making $\vec{P} \to \frac{\vec{m}}{c}$, $\vec{E} \to c\vec{B}$, $\vec{B} \to -\frac{\vec{E}}{c}$ replacements. Note that the polarization of radiation is determined by the components of $\hat{n}\times\vec{m}$.

The second integral on the right-hand side of the expression at the end of page 161 can be written as:

$$\frac{1}{2}\int \left[x'_j J_i + x'_i J_j\right] d^3x' = \frac{1}{2}\int\Big[\partial'_k\left(x'_j x'_i J_k\right) - x'_j x'_i \overbrace{\partial'_k J_k}^{\vec{\nabla}'\cdot\vec{J}}\Big]\, d^3x'$$

The first term on the right-hand side gives rise to a surface integral that, due to the source being localized, vanishes. After using $\vec{\nabla}'\cdot\vec{J}(\vec{x}') = i\omega\rho(\vec{x}')$, as shown before, we find:

$$\vec{A}_q(\vec{x}) \simeq \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(-ik)\left(-\frac{i\omega}{2}\right)\left[n_j\,\hat{e}_i\int x'_j x'_i\,\rho(\vec{x}')\,d^3x'\right]$$

But:

$$\int x'_j x'_i\,\rho(\vec{x}')\,d^3x' = \frac{1}{3}\int\left[3x'_i x'_j - r'^2\delta_{ij}\right]\rho(\vec{x}')\,d^3x' + \frac{1}{3}\delta_{ij}\int r'^2\rho(\vec{x}')\,d^3x'$$

$$= \frac{1}{3}Q_{ij} + \frac{1}{3}\delta_{ij}\int r'^2\rho(\vec{x}')\,d^3x'$$

Here $Q_{ij} \equiv \int\left[3x'_i x'_j - r'^2\delta_{ij}\right]\rho(\vec{x}')\,d^3x'$ is the ij-th component of the electric quadropole tensor of the source (hence the subscript "q" in $\vec{A}_q$). In the far-field region, therefore, we have:

$$\vec{A}_q(\vec{x}) \simeq \frac{-\mu_0 \omega k}{24\pi} \frac{e^{ikr}}{r} \left[(Q_{ij} n_j) \hat{e}_i + \hat{n} \int \rho(\vec{x}') r'^2 d^3x' \right]$$

And:

$$\vec{B}_q(\vec{x}) = \vec{\nabla} \times \vec{A}_q(\vec{x}) \simeq \frac{-i\mu_0 c k^3}{24\pi} \frac{e^{ikr}}{r} \hat{n} \times \vec{q}(\hat{n}) \quad \left(q_i(\hat{n}) \equiv \sum_{j=1}^{3} Q_{ij} n_j \right)$$

$$E_q(\vec{x}) = \frac{i}{\mu_0 \epsilon_0 \omega} \vec{\nabla} \times \vec{B}_q(\vec{x}) \simeq \frac{ick^4}{\epsilon_0 \omega} \frac{e^{ikr}}{24\pi r} \hat{n} \times (\hat{n} \times \vec{q}(\hat{n})) = \frac{ik^3}{24\pi\epsilon_0} \frac{e^{ikr}}{r} \hat{n} \times (\hat{n} \times \vec{q}(\hat{n}))$$

Thus:

$$\frac{dP_q}{d\Omega} = \frac{1}{2\mu_0} \mathrm{Re}\left[\vec{E}_q(\vec{x}) \times \vec{B}_q^*(\vec{x}) \right] \cdot \hat{n} r^2 \propto k^6 \left[|\vec{q}(\hat{n})|^2 - |\hat{n} \cdot \vec{q}(\hat{n})|^2 \right]$$

$$|\hat{n} \times \vec{q}(\hat{n})|^2 = (\hat{n} \times \vec{q}(\hat{n})) \cdot (\hat{n} \times \vec{q}^*(\hat{n})) = \vec{q}(\hat{n}) \cdot \vec{q}^*(\hat{n}) - |\hat{n} \cdot \vec{q}(\hat{n})|^2$$

In comparison with the $l=0$ term (i.e., electric dipole radiation), power in the magnetic dipole and electric quadrupole scales as $k^6 d^4$ as opposed to $k^4 d^2$ ("d" being the typical size of the source). This implies that in the long wavelength limit, when $kd \ll 1$, terms with higher l are suppressed in power of kd, thereby making the expansion a useful approximation.

We note that in cases that the electric dipole moment contribution is absent, we must consider the magnetic dipole and electric quadrupole contributions as the first non-vanishing terms. An example of such a situation is two equal charges at diametrically opposite points in uniform rotational motion on a circle.

Finally, let us comment on the general situation when the dimension of the source is not necessarily small compared to the wavelength of radiation. In this case, in the radiation zone $r \gg r', \lambda$, we have:

$$|\vec{x} - \vec{x}'| = \sqrt{r^2 + r'^2 - 2\vec{x}\cdot\vec{x}'} = r\left(1 - \frac{2\vec{x}\cdot\vec{x}'}{r^2} + \frac{r'^2}{r^2}\right)^{\frac{1}{2}} = r - \hat{n}\cdot\vec{x}' + O\left(\frac{r'^2}{r^2}\right)$$

Thus:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} e^{ik|\vec{x}-\vec{x}'|} d^3x' \simeq \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}') \frac{e^{ik(r-\hat{n}\cdot\vec{x}')}}{r-\hat{n}\cdot\vec{x}'} d^3x'$$

Now, we can neglect $\hat{n}\cdot\vec{x}'$ in the denominator as compared with r. However, we cannot do the same in the exponent of the numerator as $k\hat{n}\cdot\vec{x}'$ may still be a large phase. Hence:

$$\vec{A}(\vec{x}) \simeq \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}(\vec{x}')\, e^{-i\vec{k}\cdot\vec{x}'}\, d^3x'$$

This is just the Fourier transform of the current density! As for the $\vec{B}$ and $\vec{E}$ fields, we have:

$$\vec{B} = \vec{\nabla}\times\vec{A} \simeq ik\hat{n}\times\vec{A}$$

$$\vec{E} = \frac{ic^2}{\omega}\vec{\nabla}\times\vec{B} \simeq \frac{ic^2}{\omega}ik\hat{n}\times\vec{B} \simeq -ik\hat{n}\times(\hat{n}\times\vec{A})$$